Financ Mark Portf Manag (2009) 23: 315–327
DOI 10.1007/s11408-009-0105-2

PERSPECTIVES

A note on portfolio choice for sovereign wealth funds

Bernd Scherer

Published online: 1 July 2009

Abstract The current vast account surpluses of commodity-rich nations, combined with record account deficits in developed markets (the United States, Britain) have created a new type of investor. Sovereign wealth funds (SWF) are instrumental in deciding how these surpluses will be invested. We need to better understand the investment problem for an SWF in order to project future investment flows. Extending Gintschel and Scherer (J. Asset Manag. 9(3):215–238, 2008), we apply the portfolio choice problem for a sovereign wealth fund in a Campbell and Viceira (Strategic Asset Allocation, 2002) strategic asset allocation framework. Changing the analysis from a one to a multi-period framework allows us to establish a three-fund separation. We split the optimal portfolio for an SWF into speculative demand as well as hedge demand against oil price shocks and shocks to the short-term risk-free rate. In addition, all terms now depend on the investor's time horizon. We show that oil-rich countries should hold bonds and that the optimal investment policy for an SWF as a long-term investor is determined by long-run covariance matrices that differ from the correlation inputs that one-period (myopic) investors use.

Keywords Sovereign wealth fund · Vector autoregression · Oil price shock · Three-fund separation · Portfolio choice

JEL Classification E32 · F34 · G11

1 Introduction

The current vast account surpluses of commodity-rich nations, combined with record account deficits in developed markets (the United States, Britain), have created a new

B. Scherer (✉)
MD Morgan Stanley Investment Management & Visiting Professor Birkbeck College, London, UK
e-mail: bernd.scherer@morganstanley.com

Fig. 1 Daily oil price (Brent current month FOB) movements from January 1982 to September 2008. The underlying total wealth position of an oil-rich country can vary dramatically over time and needs management to smooth intergenerational consumption patterns. Source: Datastream

type of investor. Sovereign wealth funds (SWF) will be instrumental in deciding how these surpluses will be invested. We therefore need to better understand the investment problem for a commodity-rich investor.

For many oil-exporting countries, crude oil or gas reserves are the most important national assets. Any change in the value of reserves directly and materially affects these countries' wealth, and thus the well-being of their citizens. Figure 1 illustrates the violent nature of oil price changes, which can have a destabilizing effect on the economy via volatile real exchange rates.

In recognition of this problem, a number of oil-exporting countries have been depositing oil revenues in funds dedicated to finance future expenditures, so-called intergenerational welfare funds. However, oil-based SWFs seem to have been ill-prepared for a sudden drop in oil prices. Rather than investing in assets that pay off when oil prices reverse, their assets plunged in parallel. Setser and Ziemba (2009) estimate that the SWFs of Abu Dhabi (ADIA), Kuwait (KIA), and Quatar (QIA) lost 40% of their 2007 value.

Gintschel and Scherer (2008) identify the optimal asset allocation problem[1] for an SWF as an asset allocation problem with nontradable wealth that bears a strong resemblance to portfolio choice in the presence of human capital. A country's total wealth can be seen as a combination of financial wealth and nontradable resource (oil) wealth. According to these scholars' one-period framework, assets with negative correlation to oil wealth are well suited to improve the efficiency of total wealth for an oil-rich investor. In their empirical implementation they choose sector portfolios with little or negative correlation to oil price changes. Optimal asset allocation decisions must take these correlations into account in order to avoid welfare losses, i.e., sovereign welfare funds should integrate the management of financial and resource wealth.

Doskeland (2007) integrates the SWF into a country's budget, focusing on social system liabilities, but does not provide closed-form simulations, largely ignores oil price volatility, and does not provide a data-generating process capable of model-

[1]We use the term portfolio choice and strategic asset allocation interchangeably. See Köhler and Drobetz (2002) for the importance of strategic asset allocation.

ing mean reversion in either volatility or correlation. No closed-form solutions for portfolio choice are provided.

Scherer (2009) introduces resource uncertainty as a form of background risk, as well as the impact of optimal oil extraction, for an SWF. First, he concludes that uncertainty about the size of a country's oil wealth relative to its total wealth will make it invest less aggressively. In other words, if you do not know how rich you really are, you might want to invest less aggressively. Empirically, we should observe that SWFs with larger resource uncertainty invest less aggressively, and vice versa. Also, we would expect that economies with low reserves relative to financial wealth are less affected by resource uncertainty. Second, Scherer argues that oil wealth is dependent on the optimal path of extraction policy, which is in turn determined by the interplay between expected future oil prices, extraction costs, and the discount factor for future cash flows. He determines an exogenous extraction policy (independent of the evolution of asset returns) and investigates the impact of portfolio choice on the sequence of one-period optimization problems. As long as financial wealth is low relative to resource wealth, an SWF will need to invest aggressively in order to make a meaningful impact on total wealth. Depending on the speed of the optimal extraction policy, a maturing SWF will gradually invest less aggressively, showing less appetite for aggressive hedging or speculative risk taking.

In this paper, we extend previous work in two directions. First, we add a time dimension, which is missing from previously used one-period frameworks. Introducing a multi-period setting (i.e., an autoregressive data-generating process) with return predictability allows us to investigate how asset allocation recommendations change as the time horizon becomes longer. This is of particular relevance for SWFs as long-term investors. Campbell and Viceira (2002), for example, show that stocks are less risky in the long run, caused by mean reversion. These "term structure effects" created by the data-generating process might also have an effect on the term structure of correlation between various asset classes and oil. In other words, we investigate the term structure of hedging demand. What happens if the correlation between bonds and oil increases over time or the volatility of bonds decreases relative to the volatility of oil wealth? In both cases, we expect the optimal hedging demand to increase. Unlike previous work on SWF investing, we investigate whether a (short-) long-term investor will use a (short-) long-term measure of correlation. Second, we apply our framework to broad economic asset classes, such as equities, bonds, and listed real estate[2] in order to give observers of SWFs some idea as to the direction in which global funds are flowing. For example, if SWFs appear to view government bonds as the ultimate recession hedge, this is good news for the United States, which requires a constant inflow of capital to finance its current account deficit and stabilize its dollar.

[2]We focus on portfolio investments. This is a narrower scope of action than that possible in reality to sovereign investors. Rather than investing in securities (mostly USD dominated) abroad, sovereign investors can also use their oil revenues to attract future growth industries and develop the necessary infrastructure that will lure top human talent. Dubai and Qatar are prime examples of this. During the following exposition, we rely on the normality in return distributions assumption to allow us to come up with closed-form solutions that provide conceptual insight into the structure of the underlying problem. Although we are aware that returns on capital markets and certainly on commodities like oil are in the short run far from normal, we also believe an SWF belongs to the group of long-term investors such that the central limit theorem will aid us somewhat in mitigating the non-normality issue.

The paper proceeds as follows. Section 2 applies the portfolio choice problem to an oil-based SWF in a multi-period setting using the Campbell and Viceira (2002) framework, i.e., we introduce time into the myopic asset allocation framework discussed in the literature to date. After establishing three-fund portfolio separation, we describe our data and data-generating process in Sect. 3. Section 4 presents the term structure of optimal hedging demand, i.e., the implications of asset price dynamics for the asset allocation problem of an SWF at different time horizons. Section 5 concludes.

2 Three-fund separation: incorporating the SWF into government budgets

Following Gintschel and Scherer (2008), we view the optimal asset allocation problem of an SWF as the decision-making problem faced by an investor with nontradable endowed wealth (oil reserves). To transform those scholars' solution into the multi-period framework of Campbell and Viceira (2002), we let x_t denote our vector of asset excess returns (including oil) over cash as well as cash, and s_t be the vector of economic state variables, i.e., conditioning information. The idea here is that only unexpected variations (those not explained by state variables) pose a risk. To fix notation we denote the (conditional) annualized n period return and risk by

$$\mu^{(n)} = n^{-1} E\left[\sum_{i=1}^{n} x_{t+n} | s_t\right], \tag{1}$$

$$\Sigma^{(n)} = n^{-1} \operatorname{Var}\left(\sum_{i=1}^{n} x_{t+n} | s_t\right) \tag{2}$$

where $\mu^{(n)}$ and $\Sigma^{(n)}$ represent n period expected returns. We can also define subvectors and submatrices. To extract the required n-period risk and return measures, we use $\mu_a^{(n)}$ for the expected excess returns for assets, $\Sigma_{aa}^{(n)}$ for the covariance matrix of asset returns, $\Sigma_{ao}^{(n)}$ for the covariance vector of asset and oil returns, and $\Sigma_{ac}^{(n)}$ for the covariance of asset excess returns and short-term cash rates. Equations (1) and (2) describe the term structure of risk and return. This information is used to specify the mean reversion properties of various asset classes in order to introduce a time dimension into portfolio choice. If, for example, we find considerable mean reversion in equity returns, long-run investors could more invest more aggressively in equities than will short-term investors. Suppose a sovereign holds financial assets in an SWF as well as oil assets underground, and let $0 < \theta \leq 1$ denote the fraction of financial wealth to total wealth (oil plus financial assets).[3] Given CRRA preferences with risk-aversion parameter γ, we show that the optimal portfolio for horizon n is

[3] We assume θ to be exogenous (known). This assumption is necessary to allow tractability. An endogenous θ would require us to simultaneously determine asset mix and optimal extraction policy.

given by[4]

$$w_t^{(n)} = \frac{1}{\gamma}\left(\frac{1}{\gamma}\Sigma_{aa} + \frac{(\gamma-1)}{\gamma}\Sigma_{aa}^{(n)}\right)^{-1} \times \left[\frac{1}{\theta}\left(\mu_a^{(n)} + \frac{1}{2}\sigma_a^2\right) - (\gamma-1)\frac{(1-\theta)}{\theta}\Sigma_{ao}^{(n)} - \frac{1}{\theta}(\gamma-1)\Sigma_{ac}^{(n)}\right]. \quad (3)$$

This is equivalent to a three-fund separation.[5] The SWF wants a leveraged ($\frac{1}{\theta} > 1$) speculative portfolio that purely depends on expected asset excess returns, $w_{\text{spec.}}^{(n)}$, an equally (as changes in the risk-free rate can be managed only via the asset portfolio, even though they affect total wealth) leveraged hedging portfolio, $w_{\text{hedge,cash}}^{(n)}$, that protects against intertemporal variations in short-term rates, and another hedging portfolio that protects against oil price shocks, $w_{\text{hedge,oil}}^{(n)}$[6]

$$w_{\text{spec}}^{(n)} = \left(\frac{1}{\theta}\right)\left(\frac{1}{\gamma}\right)\left(\frac{1}{\gamma}\Sigma_{aa} + \frac{\gamma-1}{\gamma}\Sigma_{aa}^{(n)}\right)^{-1}\left(\mu^{(n)} + \frac{1}{2}\sigma_a^2\right), \quad (4)$$

$$w_{\text{hedge,cash}}^{(n)} = -\left(\frac{1}{\theta}\right)\left(\frac{\gamma-1}{\gamma}\right)\left(\frac{1}{\gamma}\Sigma_{aa} + \frac{\gamma-1}{\gamma}\Sigma_{aa}^{(n)}\right)^{-1}\Sigma_{ac}^{(n)}, \quad (5)$$

$$w_{\text{hedge,oil}}^{(n)} = -\left(\frac{1-\theta}{\theta}\right)\left(\frac{\gamma-1}{\gamma}\right)\left(\frac{1}{\gamma}\Sigma_{aa} + \frac{\gamma-1}{\gamma}\Sigma_{aa}^{(n)}\right)^{-1}\Sigma_{ao}^{(n)}. \quad (6)$$

For $\theta = 1$, we arrive at the optimal solution for an investor having financial wealth only, while for $\gamma \to \infty$, we obtain $w_{\text{hedge,cash}}^{(n)} = -(\frac{1}{\theta})[\Sigma_{aa}^{n}]^{-1}\Sigma_{ac}^{(n)}$ and $w_{\text{hedge,oil}}^{(n)} = -(\frac{1-\theta}{\theta})[\Sigma_{aa}^{(n)}]^{-1}\Sigma_{ao}^{(n)}$. It is interesting that for $\gamma = 1$, i.e., a log investor, we arrive at the familiar solution that there is no hedging demand and the optimal solution degenerates to the one-period myopic speculative demand $w_{\text{spec}}^{(n)} = \frac{1}{\theta}\Sigma_{aa}^{-1}(\mu^{(n)} + \frac{1}{2}\sigma_a^2)$. Note that this framework is extremely general and its application is not limited to equities or bonds, but can easily be applied to hedge funds, private equity investments, and the like.[7]

[4] See Appendix A for a derivation of this result.

[5] As times goes by, the investor changes the asset allocation from $w^{(n)}$ to $w^{(n-1)}$ and to $w^{(n-2)}$, etc.

[6] The optimal position in risk-free assets is given by $1 - I^T w^{(n)}$.

[7] Including investments with less data availability can be handled by using standard econometric techniques to deal with time series of different lengths and does not limit the practical usefulness of the above framework. Suppose we add a vector of new time series with shorter history, y_t. Following Stambaugh (1997), all that is needed is to "glue" the new time series to the existing VAR in the form $y_{t+1} = c + Dz_{t+1} + Ez_t + Fy_t + \eta_{t+1}$, where $\eta_{t+1} \sim N(0, \Sigma)$ and F is a diagonal matrix, i.e., the new series is allowed to exhibit autocorrelation, but it does not help forecast other series or state variables. The contemporaneous covariance is measured by D and as such we assume ε_{t+1} and η_{t+1} are uncorrelated (all common variation is caught in D). Including the new series leads us to the new VAR representation $z_{t+1}^* = \Upsilon_0 + \Upsilon_1 z_t^* + e_{t+1}$, where

$$z_{t+1}^* = \begin{bmatrix} z_{t+1} \\ y_{t+1} \end{bmatrix}, \quad \Upsilon_0 = \begin{bmatrix} a \\ c + Da \end{bmatrix}, \quad \Upsilon_1 = \begin{bmatrix} B & 0 \\ E + DB & F \end{bmatrix}, \quad \Omega^* = \begin{bmatrix} \Omega & \Omega D^T \\ D\Omega & \Sigma + D\Omega D^T \end{bmatrix}.$$

3 Return dynamics and the data-generating process

3.1 Model and data

Our model is incomplete without a specification for (1) and (2). We start with assuming a first-order vector autoregression as our data-generating process (DGP):

$$z_{t+1} = a + Bz_t + \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim N(0, \Omega) \tag{7}$$

where $z_t = [x_t \; s_t]^T$, and Ω represents the residual covariance matrix of our VAR with a vector of constants, a, and a coefficient matrix B. At this stage, concerns might arise as to the validity of (7). The residuals for the oil equation, for example, are most likely not normally distributed. However, this still would leave the coefficient estimates unbiased. In addition, we could, of course, simulate (7) as a semi-parametric VAR, i.e., by bootstrapping from the empirical residuals instead of Ω. However, over a larger number of periods, the central limit theorem will provide approximately normal variables. Finally, (7) is widely used in intertemporal portfolio choice problems. Of course, extensions are possible.[8] For example, we could estimate a cointegrating VAR in levels, rather than a stationary VAR in first differences, to capture the different speed of mean reversion and different equilibrium relations, but such would fall outside the main purpose of this paper, which is to lay out a framework. From (7) we can easily work out the multi-period expressions for risk and return in (1) and (2) as[9]

$$\Sigma^{(n)} = \sum_{i=1}^{n} \left[\left(\sum_{j=0}^{i-1} B^j \right) \Omega \left(\sum_{j=0}^{i-1} B^j \right)^T \right] \tag{8}$$

where, again, $\Sigma^{(n)}$ denotes the covariance matrix of n period returns and $B^0 = I$. Correlation, variance, and covariance can be chosen easily from the elements in $\Sigma^{(n)}$. Plotting variances, correlations, etc. against n provides a term structure of risk.[10]

We proxy the investment universe for an SWF contained in x_t by the CRSP value-weighted stock index, real estate investments (FTSE NAREIT real estate index), government bonds (Lehman Long US Treasury Bonds Index), the one-month Treasury bills rate, and real bonds (TIPS).[11] For changes in oil wealth, we use Brent current month FOB. Given very high correlations between Brent, West Texas Intermediate, and Arab Light, the choice of variable does not matter for our purpose. Economic state variables, s_t, are given by dividend yield, credit spread (difference between Baa Corporate Bond Yield and the 10-year Treasury constant maturity rate), term spread (difference between the 10-year Treasury constant maturity rate and one-month T-bill rate), and nominal yield (10-year Treasury constant maturity rate).

We can test the restrictions implicit in Υ_0, Υ_1 on an overlapping data set.

[8] See, for example, Campbell et al. (2003), Campbell et al. (2004), and Guidolin and Timmerman (2007).

[9] See Appendix B for a brief sketch.

[10] An alternative (brute force) way to calculate the covariance of multi-period returns is to simulate $z_{t+1} + z_{t+2} + \cdots + z_{t+n}$ from (7) and directly calculate the required statistics.

[11] TIPS are constructed using the methodology in Kothari and Shanken (2004).

Our analysis is based on quarterly returns from 1973 I through 2007 IV. Table 1 summarizes the data and our notation. It is noteworthy that oil returns are not statistically different from cash rates, i.e., excess returns are not statistically different from zero. This supports the Hotelling–Solow rule under perfectly integrated capital markets. Natural resource prices should grow at the world interest rate such that countries are indifferent between depletion (earning the interest rate) and keeping oil underground (earning price changes). Our state variables are among those commonly found in the literature and are motivated by time variation in investment opportunities.[12] The investment universe encompasses most of the investment opportunity available to an SWF.

3.2 Return dynamics

We start by calibrating our DGP in (7) to the data in Table 1. Coefficient estimates and R^2s are given in Table 3; Table 2 provides estimates for Ω, with the main diagonal representing quarterly volatility. Unexplained quarterly real estate volatility amounts to 6.65%, which is only marginally smaller than the unconditional volatility of 7.03%. This is hardly surprising given that the R^2 for the real estate equation is extremely low. Dividend yields have significant forecasting power for equities and bonds, and bonds can also be forecasted using the last quarter's nominal yields. Shocks on the dividend yields show negative contemporaneous correlation (−0.27) with equity returns.

Rising dividend yields has a positive impact on future equity returns (positive significant regression coefficient of 0.12). In summary, positive shocks to dividend

Table 1 Summary statistics. We report descriptive statistics for all endogenous (assets and state variables) variables in our VAR. Only the Sharpe ratios have been annualized

	Symbol	Mean (μ)	Volatility (σ)	Sharpe ($\frac{\mu-rc}{\sigma}\sqrt{4}$)	Min	Max	Skew	Kurtosis
Assets								
Oil	ro_t	−0.37	19.23	−0.04	−0.89	0.82	−0.27	6.33
Tips	rr_t	0.42	0.60	1.40	−0.02	0.02	−0.94	2.29
T-Bill	rc_t	1.29	0.59		0.00	0.03	0.32	0.12
Long Bonds	rb_t	1.33	4.97	0.54	−0.09	0.17	0.51	0.37
Equities	re_t	1.69	8.14	0.42	−0.28	0.18	−0.79	1.46
Real Estate	ri_t	1.42	7.03	0.40	−0.17	0.19	0.00	0.05
State Variables								
Nominal	yn_t	7.15	2.56		0.04	0.15	0.96	0.36
Credit Spread	cs_t	1.96	0.50		0.01	0.04	1.04	0.67
Time Spread	ts_t	1.75	1.25		−0.01	0.05	−0.05	−0.62
Dividend Yield	dy_t	−3.66	34.24		−4.55	−2.97	−0.21	−0.58

[12] See Campbell and Viceira (2002).

Table 2 Results from first-order VAR: residual covariance matrix. We report the error covariance matrix, Ω, of the VAR from (7). The main diagonal contains (quarterly) volatility while off-diagonal entries represent correlations

	rr_t	rc_t	rb_t	re_t	ro_t	ri_t	yn_t	cs_t	ts_t	dy_t
rr_t	**0.47**									
rc_t	0.05	**0.12**								
rb_t	−0.45	−0.08	**4.28**							
re_t	−0.28	−0.07	0.10	**7.11**						
ro_t	0.54	0.09	−0.22	−0.31	**17.84**					
ri_t	−0.40	−0.16	0.26	0.55	−0.28	**6.65**				
yn_t	0.20	0.08	−0.28	−0.15	−0.08	−0.17	**0.25**			
cs_t	−0.01	−0.19	0.02	−0.01	−0.01	0.01	−0.54	**0.18**		
ts_t	0.09	−0.33	−0.24	0.11	−0.21	0.02	0.47	−0.19	**0.43**	
dy_t	0.05	−0.08	0.04	−0.27	−0.02	−0.31	0.18	−0.21	0.03	**13.85**

yields (an increase) negatively impact current, but positively impact, next period returns. This creates mean reversion and is likely to reduce long-term risk. All state variables are very persistent as indicated by the high R^2 and the large highly significant autoregressive coefficients. Also note that the system of (7) is stable, as all eigenvalues of B have modulus less than 1 (the largest is 0.94). Hence the process is stable and therefore also stationary. The results confirm recent work by Driesprong et al. (2008), who found that lagged oil price changes predict future equity market returns. We find a t-value of 2.55 on quarterly data, which is higher than the t-values using weekly data.[13]

4 The term structure of hedging demand

In this section, we focus exclusively on an SWF's hedging demand, as $w_{t,\text{spec}}^{(n)}$ is essentially the same for an asset-only investor, an asset liability investor, and an asset-only investor with nontradable wealth and, hence, not specific to an SWF.[14] We ignore effects from γ and θ as they only affect leverage. The results are summarized in Figs. 2 and 3. In both cases, hedging demand is positive as long as correlation between investment and risk source is negative. In other words, assets that show negative correlation with oil or time-varying short rates will have positive weights. Correlation drives the sign of hedging demand, but it is the relative term structure of risk (does annualized volatility grow or decay with time horizon) that determines, ceteris paribus, the extent of hedging demand, which essentially equals a "beta" estimate. For example, $[\Sigma_{aa}^{(n)}]^{-1}\Sigma_{ao}^{(n)}$ describes a vector of asset betas relative to oil.

[13] Our regression coefficient shows the opposite sign but, given that their regression omitted many statistically significant explanatory variables, we should not be surprised if their results are biased.

[14] We are therefore not required to create forecasts for the asset classes involved. This topic is well covered in Drobetz (2001).

Table 3 Results from first-order VAR: parameter estimates. We report the coefficients of the VAR in (7) together with standard errors and t-values. See Table 1 for a description of individual variable names. The last two rows contain the "R-square" as well as "adjusted R-square" for each individual regression. Coefficients significant at the 5% level are presented in bold

	z_t									
	x_t						s_t			
	rr_t	rc_t	rb_t	re_t	ro_t	ri_t	yn_t	cs_t	ts_t	dy_t
a	**0.04**	−0.00	**−0.41**	**0.57**	0.62	−0.02	**0.01**	**−0.01**	**0.05**	**−1.30**
	0.01	0.00	**0.11**	**0.18**	0.46	0.17	**0.01**	**0.00**	**0.01**	**0.36**
	3.23	−0.61	**−3.73**	**3.08**	1.34	−0.12	**2.13**	**−2.02**	**4.76**	**−3.64**
rr_{t-1}	0.08	**−0.06**	**2.28**	**−4.29**	−5.77	−1.13	0.06	−0.02	0.14	−1.61
	0.13	**0.03**	**1.14**	**1.90**	4.76	1.77	0.07	0.05	0.12	3.70
	0.65	**−1.97**	**2.00**	**−2.26**	−1.21	−0.64	0.85	−0.37	1.24	−0.44
rc_{t-1}	−0.22	1.45	**−8.42**	−5.27	−7.36	**−12.23**	0.65	−0.22	**−5.65**	15.38
	0.44	0.11	**3.97**	6.59	16.53	**6.16**	0.24	0.17	**0.40**	12.83
	−0.51	13.51	**−2.12**	−0.80	−0.45	**−1.99**	2.76	−1.36	**−14.04**	1.20
rb_{t-1}	−0.02	**−0.01**	0.10	**0.42**	**−1.25**	0.13	**−0.10**	**0.03**	**−0.03**	−0.32
	0.01	**0.00**	0.11	**0.18**	**0.44**	0.16	**0.01**	**0.00**	**0.01**	0.34
	−1.51	**−5.14**	0.95	**2.43**	**−2.86**	0.78	**−16.22**	**7.52**	**−2.88**	−0.94
re_{t-1}	−0.01	0.00	0.12	0.05	−0.12	0.04	0.00	**−0.01**	0.00	−0.05
	0.01	0.00	0.07	0.11	0.29	0.11	0.00	**0.00**	0.01	0.22
	−1.71	0.10	1.75	0.45	−0.41	0.36	0.28	**−3.41**	0.71	−0.24
ro_{t-1}	−0.00	0.00	0.02	**0.12**	−0.09	0.04	0.00	−0.00	0.00	0.09
	0.00	0.00	0.03	**0.05**	0.12	0.05	0.00	0.00	0.00	0.09
	−0.71	0.75	0.52	**2.55**	−0.75	0.98	0.12	−0.19	0.54	1.01
ri_{t-1}	0.00	0.00	−0.16	−0.24	0.08	−0.01	0.01	**−0.01**	−0.01	0.04
	0.01	0.00	0.09	0.15	0.37	0.14	0.01	**0.00**	0.01	0.28
	0.50	0.67	−1.84	−1.63	0.23	−0.11	1.03	**−2.16**	−0.88	0.13
yn_{t-1}	−0.11	**−0.11**	**3.33**	−0.26	−0.96	2.71	**0.80**	0.08	**1.17**	−1.12
	0.11	**0.03**	**0.95**	1.58	3.96	1.48	**0.06**	0.04	**0.10**	3.07
	−1.05	**−4.33**	**3.51**	−0.17	−0.24	1.83	**14.13**	1.92	**12.16**	−0.37
cs_{t-1}	−0.02	**−0.07**	−1.12	−0.04	1.53	1.42	0.09	**0.82**	0.12	0.31
	0.11	**0.03**	0.99	1.65	4.13	1.54	0.06	**0.04**	0.10	3.21
	−0.18	**−2.52**	−1.13	−0.02	0.37	0.92	1.61	**19.89**	1.17	0.10
ts_{t-1}	0.06	**0.18**	−1.17	−0.80	−1.97	−1.91	**0.14**	−0.04	**−0.60**	6.74
	0.10	**0.02**	0.90	1.50	3.77	1.40	**0.05**	0.04	**0.09**	2.92
	0.63	**7.33**	−1.29	−0.53	−0.52	−1.36	**2.53**	−1.11	**−6.59**	2.31
dy_{t-1}	**0.01**	−0.00	**−0.09**	**0.12**	0.11	−0.00	**0.00**	**−0.00**	**0.01**	**0.71**
	0.00	0.00	**0.02**	**0.04**	0.10	0.04	**0.00**	**0.00**	**0.00**	**0.08**
	2.51	−0.98	**−3.63**	**2.86**	1.09	−0.07	**2.28**	**−2.88**	**4.24**	**8.86**
R^2	0.35	0.96	0.26	0.22	0.14	0.11	0.99	0.88	0.88	0.83
$\bar{R}^2$	0.28	0.96	0.18	0.14	0.04	0.01	0.99	0.86	0.86	0.81

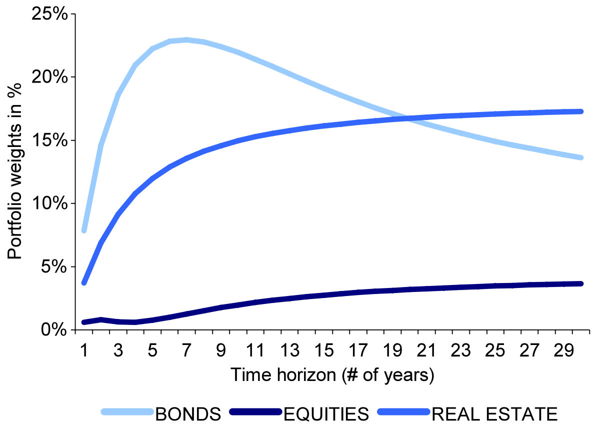

Fig. 2 Hedging time variation in short-term interest rates. We represent the hedge portfolio according to $w^{(n)}_{\text{hedge,cash}} = -\frac{1}{\theta}[\Sigma^{n}_{aa}]^{-1}\Sigma^{(n)}_{ac}$ for $n = 1, \ldots, 30$ and $\theta = \frac{1}{2}$

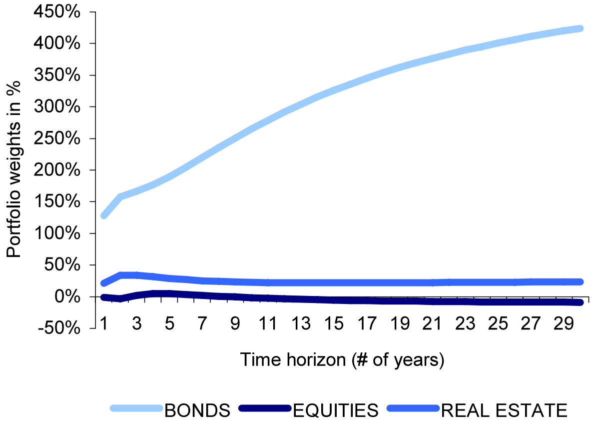

Fig. 3 Hedging oil price shocks over time. We represent the hedge portfolio according to $w^{(n)}_{\text{hedge,oil}} = -(\frac{1-\theta}{\theta}) \times [\Sigma^{(n)}_{aa}]^{-1}\Sigma^{(n)}_{ao}$ for $n = 1, \ldots, 30$ and $\theta = \frac{1}{2}$

Long government bonds play the biggest role in both hedge portfolios as government bonds are both a recession hedge (they increase in value when oil prices fall during a recession) as well as a hedge against deteriorating investment opportunities, i.e., falling rates. Although any empirical result is subject to estimation error,[15] we observe that the sign of our hedging demand coincides with our economic prior. Government bonds are a recession hedge and tend to pay off in those states of the world where demand for commodities is low and, hence, prices are depressed. Even though our data set stops at the end of 2007, we believe that this hedge would have worked extremely well in 2008, when all asset classes experienced record losses and oil fell from $150 to below $40 a barrel. Only government bonds performed well during this period, with the 10-year Treasury note up about 20%.

[15] See Fabozzi et al. (2008) for an application of Gibbs sampling on models like (7).

5 Conclusion

Our paper's unique contribution to the literature is an extension of the portfolio choice problem for oil-rich sovereign investors from a one-period to a multi-period model. This results in separate terms for speculative demand as well as for hedging demand in an environment of intertemporal variation in short rates and shocks to oil wealth. We arrive at a three-fund separation; in addition, all terms now depend on the investor's time horizon. This contrasts with earlier work that ignored the impact of mean reversion on portfolio choice for an SWF. We also provide an empirical illustration of our framework that includes equities, bonds, and listed real estate. Our analysis leads us to conclude that an SWF should hold a considerable amount of assets in long U.S. government bonds in order to hedge against the negative effects of oil price shocks as well as against deteriorating short rates. Empirically, our results are supported by the fact that part of the growing current account surplus of commodity exporters has been invested in U.S. Treasury bonds. However, with SWF funds recording losses of about 40% in 2008, while government bonds yielded about 20% in the same period, the typical SWF did not allocate nearly enough to bonds.

Acknowledgements The paper benefitted greatly from the advice and comments of an anonymous referee.

Appendix A

We can write the n period return, $\mu_{t,\mathrm{SWF}}^{(n)}$, and risk, $\sigma_{t,\mathrm{SWF}}^{(n)2}$, as a function of our decision vector of portfolio weights, $w_t^{(n)}$, according to

$$\mu_{t,\mathrm{SWF}}^{(n)} = \theta w_t^{(n)T}\left(\mu_a^{(n)} + \frac{1}{2}\sigma_a^2\right) - \frac{1}{2}\theta w_t^{(n)T}\Sigma_{aa}\theta w_t^{(n)}, \tag{A.1}$$

$$\sigma_{t,\mathrm{SWF}}^{(n)2} = \theta^2 w_t^{(n)T}\Sigma_{aa}^{(n)} w_t + 2\theta(1-\theta) w_t^{(n)T}\Sigma_{ao}^{(n)} + 2\theta w_t^{(n)T}\Sigma_{ac}^{(n)}. \tag{A.2}$$

Note that we (in accordance with the literature) implicitly assume $w_t^{(n)}$ to remain constant, which equates to assuming an investor with a constant time horizon n. Assuming CRRA, we maximize $\frac{1}{1-\gamma}(W_{\mathrm{SWF},t+n})^{1-\gamma}$ and the optimization problem becomes

$$\max_{w_t^{(n)}}\left[\mu_{t,\mathrm{SWF}}^{(n)} + \frac{1}{2}(1-\gamma)\sigma_{t,\mathrm{SWF}}^{(n)2}\right]. \tag{A.3}$$

Differentiating (A.3), we obtain the first-order conditions

$$\theta\left(\mu_a^{(n)} + \frac{1}{2}\sigma_a^2\right) - \theta^2\Sigma_{aa}w_t^{(n)} + \frac{(1-\gamma)}{2}\theta^2 2\Sigma_{aa}^{(n)}w_t^{(n)} + \frac{(1-\gamma)}{2}2\theta(1-\theta)\Sigma_{ao}^{(n)} + \frac{(1-\gamma)}{2}2\theta\Sigma_{ac}^{(n)} = 0. \tag{A.4}$$

We divide by θ, collect all terms involving $w_t^{(n)}$ on the left side, and use $(1-\gamma) = -(\gamma - 1)$ to finally arrive at

$$w_t^{(n)} = \frac{1}{\gamma}\left(\frac{1}{\gamma}\Sigma_{aa} + \frac{(\gamma-1)}{\gamma}\Sigma_{aa}^{(n)}\right)^{-1} \times \left[\frac{1}{\theta}\left(\mu_a^{(n)} + \frac{1}{2}\sigma_a^2\right) - (\gamma-1)\frac{(1-\theta)}{\theta}\Sigma_{ao}^{(n)} - \frac{1}{\theta}(\gamma-1)\Sigma_{ac}^{(n)}\right] \quad \text{(A.5)}$$

which is identical to (A.3) in the main text.

Appendix B

In this appendix, we elaborate on (8) in the main text. For illustrative purposes, we first present a two-period example, which we will generalize later. The two-period returns from a first-order VAR are given by

$$\begin{aligned} z_{t+1} + z_{t+2} &= a + Bz_t + \varepsilon_{t+1} + a + Bz_{t+1} + \varepsilon_{t+2} \\ &= a + Bz_t + \varepsilon_{t+1} + a + B(a + Bz_t + \varepsilon_{t+1}) + \varepsilon_{t+2} \\ &= a + Bz_t + \varepsilon_{t+1} + a + BA + B^2 z_t + B\varepsilon_{t+1} + \varepsilon_{t+2}. \end{aligned} \quad \text{(B.1)}$$

Uncertainty arises only from the residual terms as the model parameters are assumed to be fully known in t (no estimation error) as well as z_t. We then obtain

$$\begin{aligned} \Sigma^{(2)} &= \mathrm{Var}(z_{t+1} + z_{t+2}) = \mathrm{Var}(\varepsilon_{t+1} + B\varepsilon_{t+1} + \varepsilon_{t+2}) \\ &= \mathrm{Var}(\varepsilon_{t+1}(I + B) + \varepsilon_{t+2}) \\ &= (I + B)\Omega(I + B)^T + \Omega. \end{aligned} \quad \text{(B.2)}$$

The n period generalization follows the same logic as (2) and can be calculated via the following expression:

$$\begin{aligned} \Sigma^{(n)} &= \Omega \\ &\quad + (I + B)\Omega(I + B)^T \\ &\quad + (I + B + B^2)\Omega(I + B + B^2)^T \\ &\quad + \cdots \\ &\quad + \left(I + B + B^2 + \cdots + B^{n-1}\right)\Omega(I + B + B^2 + \cdots + B^{n-1})^T \\ &= \sum_{i=1}^{n}\left[\left(\sum_{j=0}^{i-1} B^j\right)\Omega\left(\sum_{j=0}^{i-1} B^j\right)^T\right] \end{aligned} \quad \text{(B.3)}$$

which is exactly what we find in the main text.

References

Campbell, J., Viceira, L.: Strategic Asset Allocation. Oxford University Press, Oxford (2002)
Campbell, J., Chan, Y., Viceira, L.: A multivariate model for strategic asset allocation. J. Financ. Econ. **67**, 41–80 (2003)
Campbell, J., Chako, G., Rodriquez, J., Viceira, L.: Strategic asset allocation in a continuous time VAR model. J. Econ. Dyn. Control **28**, 2195–2214 (2004)
Doskeland, T.: Strategic asset allocation for a country: the Norwegian case. Financ. Mark. Portf. Manag. **21**(2), 167–201 (2007)
Driesprong, G., Jacobsen, B., Maat, B.: Striking oil: another puzzle? J. Financ. Econ. **89**(2), 307–327 (2008)
Drobetz, W.: How to avoid the pitfalls in portfolio optimization? Putting the Black–Litterman approach at work. J. Financ. Mark. Portf. Manag. **15**(1), 59–75 (2001)
Fabozzi, F., Rachev, S., Hsu, J., Bagasheva, B.: Bayesian Methods in Finance. Wiley, New York (2008)
Gintschel, A., Scherer, B.: Optimal asset allocation for sovereign wealth funds. J. Asset Manag. **9**(3), 215–238 (2008)
Guidolin, M., Timmerman, A.: Strategic asset allocation and consumption under multivariate regime switching. J. Econ. Dyn. Control **31**, 3503–3544 (2007)
Köhler, F., Drobetz, W.: The contribution of asset allocation policy to portfolio performance. Financ. Mark. Portf. Manag. **16**(2), 219–233 (2002)
Kothari, S., Shanken, J.: Asset allocation with inflation protected bonds. Financ. Anal. J. **60**(1), 45–70 (2004)
Scherer, B.: Portfolio choice for oil-based sovereign wealth funds. In: BIS/World Bank/ECB Conference Proceedings on SAA for Central Banks and Sovereign Wealth Funds (2009)
Setser, B., Ziemba, R.: Sovereign funds—reversal of fortune. CGS Working Paper (2009)
Stambaugh, R.: Analysing investments whose histories differ in length. J. Financ. Econ. **54**, 375–421 (1997)

Bernd Scherer is Managing Director at Morgan Stanley Investment Management. He authored, co-authored or edited 7 books on quantitative asset management and published in Financial Markets and Portfolio Management, Journal of Economics and Statistics, Journal of Money Banking and Finance, Financial Analysts Journal, Journal of Portfolio Management, Journal of Investment Management, Risk, Journal of Applied Corporate Finance, Journal of Asset Management, etc. He received Master's degrees in Economics from the University of Augsburg, and the University of London and a Ph.D. from the University of Giessen. Bernd is currently Visiting Professor at Birkbeck College, London.